Lecture No. 10

- Let us examine Global to Local basis function relationship to explain how global assembly works by starting with global or cardinal bases and then deconstructing them into local bases split into elemental components

- Consider the example problem

$$\frac{d^2u}{dx^2} = p(x) \qquad 0 \le x \le 1$$

$$u(x = 0) = A$$

$$u(x = 1) = B$$

- Develop a weak weighted residual formulation

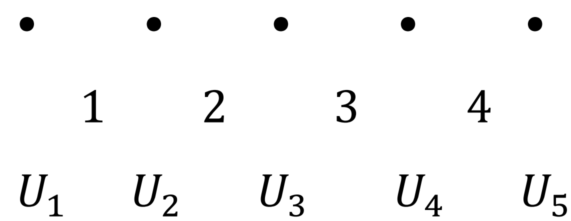
$$\langle \frac{d^2u_{app}}{dx^2}, w_j \rangle_\Omega - \langle p(x), w_j \rangle_\Omega = 0$$

$$\langle -\frac{du_{app}}{dx}, \frac{dw_j}{dx} \rangle_\Omega + \langle \frac{du_{app}}{dx}, w_j \rangle_\Gamma - \langle p(x), w_j \rangle_\Omega = 0$$

However $w_j\big|_\Gamma = 0$ (since all boundaries are essential)

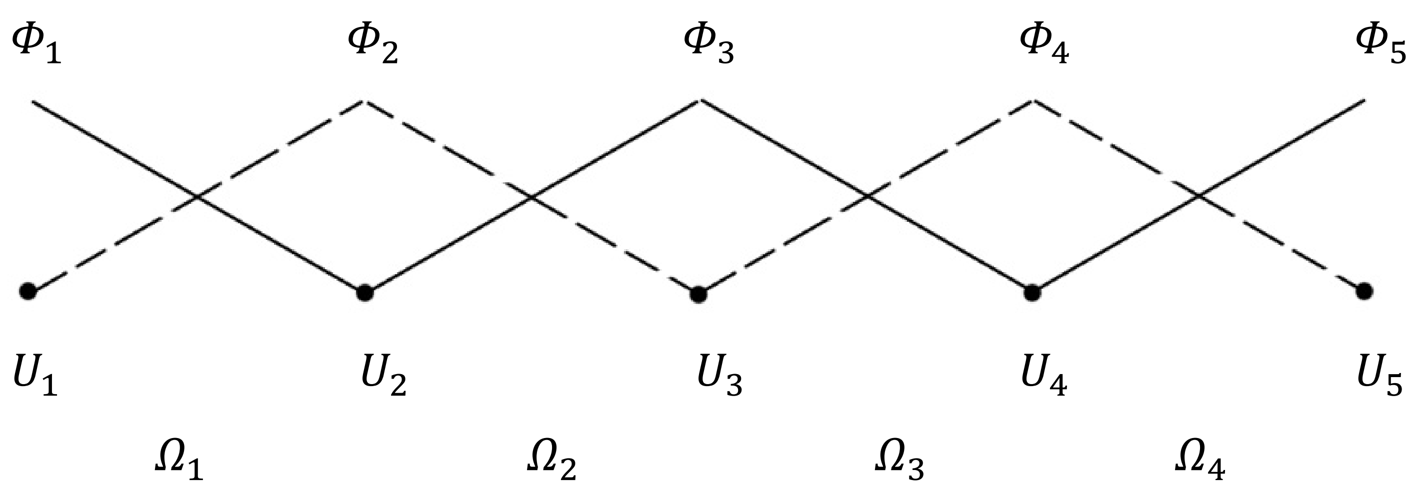
$$\langle -\frac{du_{app}}{dx}, \frac{dw_j}{dx} \rangle_\Omega - \langle p(x), w_j \rangle_\Omega = 0$$

- Divide the domain into 4 elements



With a global approximation using cardinal bases (i.e. globally defined trial functions)

$$u_{app} = U_1\Phi_1 + U_2\Phi_2 + U_3\Phi_3 + U_4\Phi_4 + U_5\Phi_5$$



Let's put off enforcing the b.c.'s until the very end (since we know we can set $U_1 = A$ and $U_5 = B$)

- Thus we have 5 unknowns coefficients (U_1 through U_5), 5 globally defined trial functions (Φ_1 through Φ_5) and 5 globally defined weighting or test functions which for Galerkin are defined as

$$w_1 = \Phi_1$$

$$w_2 = \Phi_2$$

$$w_3 = \Phi_3$$

$$w_4 = \Phi_4$$

$$w_5 = \Phi_5$$

- Substituting into the symmetrical weak weighted residual statement

$$\langle -\frac{d}{dx}(U_1\Phi_1 + U_2\Phi_2 + U_3\Phi_3 + U_4\Phi_4 + U_5\Phi_5), \frac{d\Phi_j}{dx}\rangle_\Omega - \langle p(x), \Phi_j\rangle_\Omega = 0 \quad j = 1,5$$

$\Rightarrow$

$$\langle -\left(U_1\frac{d\Phi_1}{dx} + U_2\frac{d\Phi_2}{dx} + U_3\frac{d\Phi_3}{dx} + U_4\frac{d\Phi_4}{dx} + U_5\frac{d\Phi_5}{dx}\right), \frac{d\Phi_j}{dx}\rangle_\Omega - \langle p(x), \Phi_j\rangle_\Omega = 0 \quad j = 1,5$$

- For $j = 1$

$$\langle -\left(U_1 \frac{d\Phi_1}{dx} + U_2 \frac{d\Phi_2}{dx} + U_3 \frac{d\Phi_3}{dx} + U_4 \frac{d\Phi_4}{dx} + U_5 \frac{d\Phi_5}{dx}\right), \frac{d\Phi_1}{dx}\rangle_{\Omega} - \langle p(x), \Phi_1\rangle_{\Omega} = 0$$

Since Φ_1 is non-zero only in element 1, Ω_1

$$\langle -\frac{d\Phi_1}{dx}, \frac{d\Phi_1}{dx}\rangle_{\Omega_1} U_1 + \langle -\frac{d\Phi_2}{dx}, \frac{d\Phi_1}{dx}\rangle_{\Omega_1} U_2 = \langle p(x), \Phi_1\rangle_{\Omega_1}$$

- For $j = 2$ (since Φ_2 is non-zero only in elements 1 and 2)

$$\langle -\left(U_1 \frac{d\Phi_1}{dx} + U_2 \frac{d\Phi_2}{dx} + U_3 \frac{d\Phi_3}{dx}\right), \frac{d\Phi_2}{dx}\rangle_{\Omega_1+\Omega_2} - \langle p(x), \Phi_2\rangle_{\Omega_1+\Omega_2} = 0$$

Further noting that Φ_1 is only nonzero in element 1 and Φ_3 is zero in element 1

$$\langle -\frac{d\Phi_1}{dx}, \frac{d\Phi_2}{dx}\rangle_{\Omega_1} U_1 + \langle -\frac{d\Phi_2}{dx}, \frac{d\Phi_2}{dx}\rangle_{\Omega_1+\Omega_2} U_2 + \langle -\frac{d\Phi_3}{dx}, \frac{d\Phi_2}{dx}\rangle_{\Omega_2} U_3 = \langle p(x), \Phi_2\rangle_{\Omega_1+\Omega_2}$$

- For $j=3$ and since Φ_3 is non-zero only in elements 2 and 3

$$\langle -\left(U_2\frac{d\Phi_2}{dx}+U_3\frac{d\Phi_3}{dx}+U_4\frac{d\Phi_4}{dx}\right),\frac{d\Phi_3}{dx}\rangle_{\Omega_2+\Omega_3}-\langle p(x),\Phi_3\rangle_{\Omega_2+\Omega_3}=0$$

Φ_2 is non-zero only in elements 1 and 2

Φ_3 is non-zero only in elements 2 and 3

Φ_4 is non-zero only in elements 3 and 4

$$\langle -\frac{d\Phi_2}{dx},\frac{d\Phi_3}{dx}\rangle_{\Omega_2}U_2+\langle -\frac{d\Phi_3}{dx},\frac{d\Phi_3}{dx}\rangle_{\Omega_2+\Omega_3}U_3+\langle -\frac{d\Phi_4}{dx},\frac{d\Phi_3}{dx}\rangle_{\Omega_3}U_4=\langle p(x),\Phi_3\rangle_{\Omega_2+\Omega_3}$$

- For $j=4$ and since Φ_4 is non-zero only in Ω_3 and Ω_4

$$\langle -\frac{d}{dx}(U_3\Phi_3+U_4\Phi_4+U_5\Phi_5),\frac{d\Phi_4}{dx}\rangle_{\Omega_3+\Omega_4}-\langle p(x),\Phi_4\rangle_{\Omega_3+\Omega_4}=0$$

Φ_3 is non-zero only in Ω_2 and Ω_3

Φ_4 is non-zero only in Ω_3 and Ω_4

Φ_5 is non-zero only in Ω_4

$$\langle -\frac{d\Phi_3}{dx},\frac{d\Phi_4}{dx}\rangle_{\Omega_3}U_3+\langle -\frac{d\Phi_4}{dx},\frac{d\Phi_4}{dx}\rangle_{\Omega_3+\Omega_4}U_4+\langle -\frac{d\Phi_5}{dx},\frac{d\Phi_4}{dx}\rangle_{\Omega_4}U_5=\langle p(x),\Phi_4\rangle_{\Omega_3+\Omega_4}$$

- For $j = 5$ and noting that Φ_5 is non-zero only in Ω_4

$$\langle -\frac{d}{dx}(U_4\Phi_4 + U_5\Phi_5), \frac{d\Phi_5}{dx}\rangle_{\Omega_4} - \langle p(x), \Phi_5\rangle_{\Omega_4} = 0$$

$$\langle -\frac{d\Phi_4}{dx}, \frac{d\Phi_5}{dx}\rangle_{\Omega_4} U_4 + \langle -\frac{d\Phi_5}{dx}, \frac{d\Phi_5}{dx}\rangle_{\Omega_4} U_5 = \langle p(x), \Phi_5\rangle_{\Omega_4}$$

These 5 equations can readily be assembled into a global system

$$\begin{bmatrix} \left\langle -\frac{d\Phi_1}{dx},\frac{d\Phi_1}{dx}\right\rangle_{\Omega_1} & \left\langle -\frac{d\Phi_2}{dx},\frac{d\Phi_1}{dx}\right\rangle_{\Omega_1} & & & \\ \left\langle -\frac{d\Phi_1}{dx},\frac{d\Phi_2}{dx}\right\rangle_{\Omega_1} & \left\langle -\frac{d\Phi_2}{dx},\frac{d\Phi_2}{dx}\right\rangle_{\Omega_1+\Omega_2} & \left\langle -\frac{d\Phi_3}{dx},\frac{d\Phi_2}{dx}\right\rangle_{\Omega_2} & & \\ & \left\langle -\frac{d\Phi_2}{dx},\frac{d\Phi_3}{dx}\right\rangle_{\Omega_2} & \left\langle -\frac{d\Phi_3}{dx},\frac{d\Phi_3}{dx}\right\rangle_{\Omega_2+\Omega_3} & \left\langle -\frac{d\Phi_4}{dx},\frac{d\Phi_3}{dx}\right\rangle_{\Omega_3} & \\ & & \left\langle -\frac{d\Phi_3}{dx},\frac{d\Phi_4}{dx}\right\rangle_{\Omega_3} & \left\langle -\frac{d\Phi_4}{dx},\frac{d\Phi_4}{dx}\right\rangle_{\Omega_3+\Omega_4} & \left\langle -\frac{d\Phi_5}{dx},\frac{d\Phi_4}{dx}\right\rangle_{\Omega_4} \\ & & & \left\langle -\frac{d\Phi_4}{dx},\frac{d\Phi_5}{dx}\right\rangle_{\Omega_4} & \left\langle -\frac{d\Phi_5}{dx},\frac{d\Phi_5}{dx}\right\rangle_{\Omega_4} \end{bmatrix}$$

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} \langle p(x), \Phi_1 \rangle_{\Omega_1} \\ \langle p(x), \Phi_2 \rangle_{\Omega_1+\Omega_2} \\ \langle p(x), \Phi_3 \rangle_{\Omega_2+\Omega_3} \\ \langle p(x), \Phi_4 \rangle_{\Omega_3+\Omega_4} \\ \langle p(x), \Phi_5 \rangle_{\Omega_4} \end{bmatrix}$$

- We note that to enforce b.c.'s globally we simply set $U_1 = A$ and $U_5 = B \Rightarrow W_1 = 0, W_5 = 0$ eliminating the first and last equation which we replace with

$$1.0 * U_1 = A \quad \text{and} \quad 1.0 * U_5 = B$$

- Prior to enforcing the boundary conditions we have 5 equations and 5 unknowns

- However we can also split the global system all into elemental systems

$$\begin{bmatrix} \langle -\frac{d\Phi_1}{dx}, \frac{d\Phi_1}{dx} \rangle_{\Omega_1} & \langle -\frac{d\Phi_2}{dx}, \frac{d\Phi_1}{dx} \rangle_{\Omega_1} \\ \langle -\frac{d\Phi_1}{dx}, \frac{d\Phi_2}{dx} \rangle_{\Omega_1} & \langle -\frac{d\Phi_2}{dx}, \frac{d\Phi_2}{dx} \rangle_{\Omega_1} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} \langle p(x), \Phi_1 \rangle_{\Omega_1} \\ \langle p(x), \Phi_2 \rangle_{\Omega_1} \end{bmatrix}$$

$$+$$

$$\begin{bmatrix} \langle -\frac{d\Phi_2}{dx}, \frac{d\Phi_2}{dx} \rangle_{\Omega_2} & \langle -\frac{d\Phi_3}{dx}, \frac{d\Phi_2}{dx} \rangle_{\Omega_2} \\ \langle -\frac{d\Phi_2}{dx}, \frac{d\Phi_3}{dx} \rangle_{\Omega_2} & \langle -\frac{d\Phi_3}{dx}, \frac{d\Phi_3}{dx} \rangle_{\Omega_2} \end{bmatrix} \begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} \langle p(x), \Phi_2 \rangle_{\Omega_2} \\ \langle p(x), \Phi_3 \rangle_{\Omega_2} \end{bmatrix}$$

$$+$$

$$\begin{bmatrix} \langle -\frac{d\Phi_3}{dx}, \frac{d\Phi_3}{dx} \rangle_{\Omega_3} & \langle -\frac{d\Phi_4}{dx}, \frac{d\Phi_3}{dx} \rangle_{\Omega_3} \\ \langle -\frac{d\Phi_3}{dx}, \frac{d\Phi_4}{dx} \rangle_{\Omega_3} & \langle -\frac{d\Phi_4}{dx}, \frac{d\Phi_4}{dx} \rangle_{\Omega_3} \end{bmatrix} \begin{bmatrix} U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} \langle p(x), \Phi_3 \rangle_{\Omega_3} \\ \langle p(x), \Phi_4 \rangle_{\Omega_3} \end{bmatrix}$$

$$+$$

$$\begin{bmatrix} \langle -\frac{d\Phi_4}{dx}, \frac{d\Phi_4}{dx} \rangle_{\Omega_4} & \langle -\frac{d\Phi_5}{dx}, \frac{d\Phi_4}{dx} \rangle_{\Omega_4} \\ \langle -\frac{d\Phi_4}{dx}, \frac{d\Phi_5}{dx} \rangle_{\Omega_4} & \langle -\frac{d\Phi_5}{dx}, \frac{d\Phi_5}{dx} \rangle_{\Omega_4} \end{bmatrix} \begin{bmatrix} U_4 \\ U_5 \end{bmatrix} = \begin{bmatrix} \langle p(x), \Phi_4 \rangle_{\Omega_4} \\ \langle p(x), \Phi_5 \rangle_{\Omega_4} \end{bmatrix}$$

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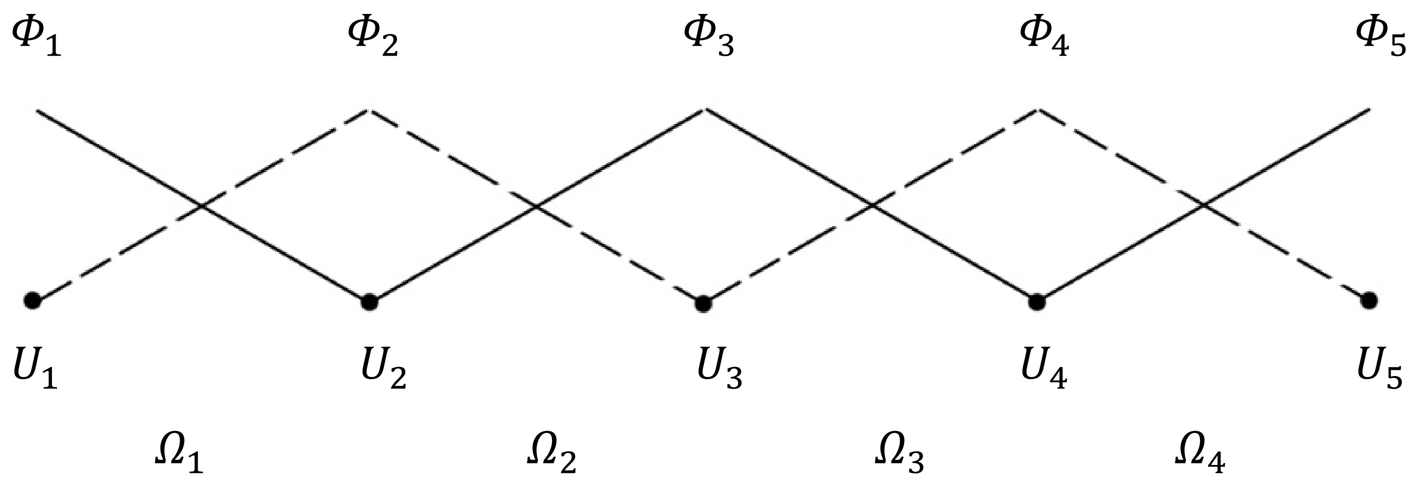

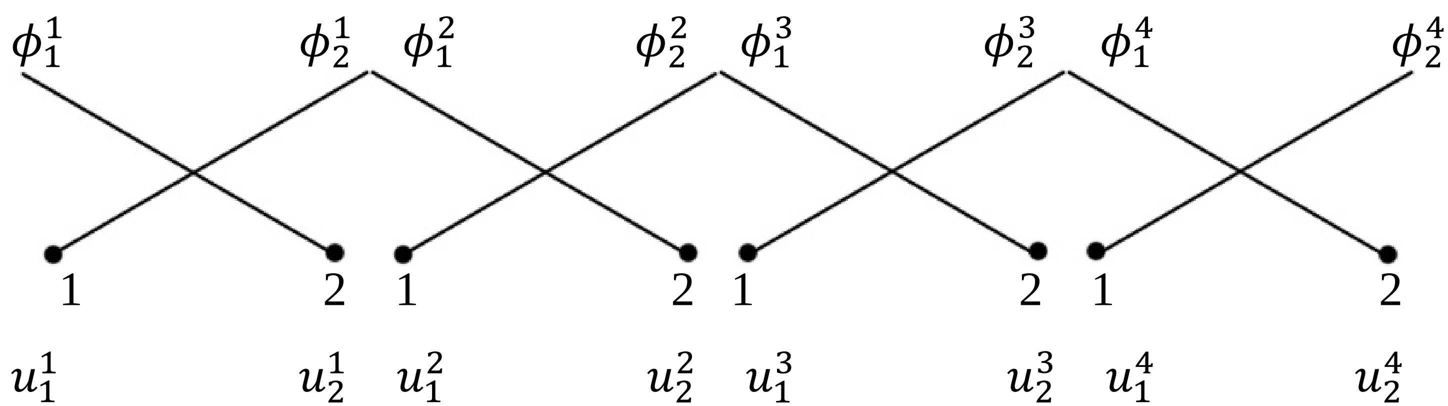

in Ω_1	$U_1 \rightarrow u_1^1$	in Ω_2	$U_2 \rightarrow u_1^2$	in Ω_3	$U_3 \rightarrow u_1^3$	in Ω_4	$U_4 \rightarrow u_1^4$
	$U_2 \rightarrow u_2^1$		$U_3 \rightarrow u_2^2$		$U_4 \rightarrow u_2^3$		$U_5 \rightarrow u_2^4$
	$\Phi_1 \rightarrow \phi_1^1$		$\Phi_2 \rightarrow \phi_1^2$		$\Phi_3 \rightarrow \phi_1^3$		$\Phi_4 \rightarrow \phi_1^4$
	$\Phi_2 \rightarrow \phi_2^1$		$\Phi_3 \rightarrow \phi_2^2$		$\Phi_4 \rightarrow \phi_2^3$		$\Phi_5 \rightarrow \phi_2^4$

Now the systems can be expressed locally as

$$\begin{bmatrix} \langle -\frac{d\phi_1^1}{dx}, \frac{d\phi_1^1}{dx} \rangle_{\Omega_1} & \langle -\frac{d\phi_2^1}{dx}, \frac{d\phi_1^1}{dx} \rangle_{\Omega_1} \\ \langle -\frac{d\phi_1^1}{dx}, \frac{d\phi_2^1}{dx} \rangle_{\Omega_1} & \langle -\frac{d\phi_2^1}{dx}, \frac{d\phi_2^1}{dx} \rangle_{\Omega_1} \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \end{bmatrix} = \begin{bmatrix} \langle p(x), \phi_1^1 \rangle_{\Omega_1} \\ \langle p(x), \phi_2^1 \rangle_{\Omega_1} \end{bmatrix}$$

$$+$$

$$\begin{bmatrix} \langle -\frac{d\phi_1^2}{dx}, \frac{d\phi_1^2}{dx} \rangle_{\Omega_2} & \langle -\frac{d\phi_2^2}{dx}, \frac{d\phi_1^2}{dx} \rangle_{\Omega_2} \\ \langle -\frac{d\phi_1^2}{dx}, \frac{d\phi_2^2}{dx} \rangle_{\Omega_2} & \langle -\frac{d\phi_2^2}{dx}, \frac{d\phi_2^2}{dx} \rangle_{\Omega_2} \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_2^2 \end{bmatrix} = \begin{bmatrix} \langle p(x), \phi_1^2 \rangle_{\Omega_2} \\ \langle p(x), \phi_2^2 \rangle_{\Omega_2} \end{bmatrix}$$

$$+$$

$$\begin{bmatrix} \langle -\frac{d\phi_1^3}{dx}, \frac{d\phi_1^3}{dx} \rangle_{\Omega_3} & \langle -\frac{d\phi_2^3}{dx}, \frac{d\phi_1^3}{dx} \rangle_{\Omega_3} \\ \langle -\frac{d\phi_1^3}{dx}, \frac{d\phi_2^3}{dx} \rangle_{\Omega_3} & \langle -\frac{d\phi_2^3}{dx}, \frac{d\phi_2^2}{dx} \rangle_{\Omega_3} \end{bmatrix} \begin{bmatrix} u_1^3 \\ u_2^3 \end{bmatrix} = \begin{bmatrix} \langle p(x), \phi_1^3 \rangle_{\Omega_3} \\ \langle p(x), \phi_2^3 \rangle_{\Omega_3} \end{bmatrix}$$

$$+$$

$$\begin{bmatrix} \langle -\frac{d\phi_1^4}{dx}, \frac{d\phi_1^4}{dx} \rangle_{\Omega_4} & \langle -\frac{d\phi_2^4}{dx}, \frac{d\phi_1^4}{dx} \rangle_{\Omega_4} \\ \langle -\frac{d\phi_1^4}{dx}, \frac{d\phi_2^4}{dx} \rangle_{\Omega_4} & \langle -\frac{d\phi_2^4}{dx}, \frac{d\phi_2^4}{dx} \rangle_{\Omega_4} \end{bmatrix} \begin{bmatrix} u_1^4 \\ u_2^4 \end{bmatrix} = \begin{bmatrix} \langle p(x), \phi_1^4 \rangle_{\Omega_4} \\ \langle p(x), \phi_2^4 \rangle_{\Omega_4} \end{bmatrix}$$

- Thus we can conclude that
  - ✓ The local problems can be assembled into a global summation. This summed set of local problems is identical to the original global expansions!!
  - ✓ Boundary conditions are implemented as a last step
  - ✓ Each element is expanded locally, then assembled locally
  - ✓ Note that if we transform each local element into a unit element all basis functions look exactly alike once we are operating in that element